

Functions with a Mean Value Property

LEOPOLD FLATTO

Communicated by J. L. WALSH

1. Introduction. It is well known that a harmonic function $f(x_1, x_2, \dots, x_n)$ in n variables in a given region R may be characterized by the arithmetic mean properties, namely

$$1.1) \quad f(x_0) = \frac{1}{V} \int_K f(x) d\omega, \quad x_0 \in R$$

$$1.2) \quad f(x_0) = \frac{1}{A} \int_S f(x) d\sigma.$$

Here K, S denote any sphere and its surface with x_0 for center which lies in R . $d\omega$ and $d\sigma$ stand for the usual Euclidean measures of K and S , V and A for the total measures of K and S . We use the letters x, y to denote the n -dimensional vectors $(x_1, \dots, x_n), (y_1, \dots, y_n)$. (For a proof of the above statement, see [4] p. 249.)

One may try to generalize 1.1) and 1.2) in two directions. We can either (I) replace K and S by other figures or (II) replace the Euclidean measures by other ones. A number of authors have dealt with generalization I [2, 3, 7]; their work is restricted to the two-dimensional case. In this paper we treat both generalizations for the case of n dimensions.

We will show in section 2 that in general the function f will satisfy a second order elliptic equation with constant coefficients. Using this information we proceed to show in section 3 that the function f must furthermore satisfy a set of auxiliary partial differential equations. Finally, in section 4, we discuss the conditions under which the solution space of these equations is finite dimensional. It turns out that the case $n = 2$ is somewhat simpler than the higher dimensional ones. The case $n = 1$ is almost trivial; it will follow from section 2 that the solutions are either constant or linear functions.

2. The elliptic equation. We prove the following theorem which generalizes a result of [3]. Without any loss of generality we assume $f(x)$ to be real valued.

Theorem 2.1. *Let $f(x) \in C$ for $x \in R$. Let Y be a fixed bounded set and let μ be a measure on Y with $\mu(Y) = 1$. (We assume that functions which are continuous*