

# *Asymptotic Distribution of the Eigenvalues of the Lower Part of the Schrödinger Operator Spectrum*<sup>1</sup>

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**1. Introduction.** Let  $V(\mathbf{x})$  be a given, real valued, measurable function over  $\mathbf{x} \in E_n$ , euclidean  $n$ -space. In the HILBERT space  $L_2(E_n)$  of finitely square integrable complex functions with respect to LEBESGUE measure  $\mu_n$  over  $E_n$ , consider the symmetric SCHRÖDINGER operator  $\mathbf{H}_0$  defined by

$$(1.1) \quad [\mathbf{H}_0 u](\mathbf{x}) = -\nabla^2 u(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x})$$

for  $u \in \mathfrak{D}_0$ , where

$$\nabla^2 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

is the Laplacian and where  $\mathfrak{D}_0 \subseteq L_2(E_n)$  is the set of all complex valued functions  $u$  over  $E_n$  which possess continuous second partials everywhere, which have  $V(\mathbf{x})u(\mathbf{x})$  to be in  $L_2(E_n)$ , and such that  $u$  and all its partials of order  $\leq 2$  are  $O([1 + |\mathbf{x}|^m] \exp(-\frac{1}{2}|\mathbf{x}|^2))$  over  $\mathbf{x} \in E_n$  for some  $m \geq 0$  depending on  $u$ .

Let  $\mathbf{H}$  be a suitable self-adjoint extension of  $\mathbf{H}_0$ , as specified more precisely in what follows; let  $\mathbf{E}(A)$  be the corresponding spectral measure; and let  $N(\lambda)$  for real  $\lambda$  be the dimension of the  $L_2(E_n)$  subspace  $\mathbf{E}((-\infty, \lambda]) L_2(E_n)$ . Thus if  $N(\lambda)$  is finite, then  $(-\infty, \lambda]$  contains only point spectra of  $\mathbf{H}$  and  $N(\lambda)$  is the number of these eigenvalues  $\lambda' \leq \lambda$ , repeated according to multiplicity.

Now consider the formula

$$(1.2) \quad N(\lambda) = \{1 + o(1)\} \left[ (2\sqrt{\pi})^n \Gamma\left(\frac{n}{2} + 1\right) \right]^{-1} \int_{\{\mathbf{x} | V(\mathbf{x}) \leq \lambda\}} [\lambda - V(\mathbf{x})]^{n/2} d\mu_n(\mathbf{x}).$$

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