

Commutators of Unitary Matrices Which Commute with One Factor

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1. Introduction. Additive commutators of matrices which commute with one factor have been studied a great deal recently. The multiplicative case plays an important role in group theory and a classical result is the following theorem (Theorem 197 in SPEISER [1], based on results of BIEBERBACH [2] and FROBENIUS [3]):

Theorem 1. *Let $C = ABA^{-1}B^{-1}$ be the commutator of the two unitary $n \times n$ matrices A and B . Assume that the characteristic roots of B are included in an arc less than a semi-circle. If A commutes with C , then A commutes with B .*

The question then arises: What is the structure of B if A does not commute with B ? The following theorem provides an answer to this question.

Theorem 2. *Let $C = ABA^{-1}B^{-1}$ be the commutator of the two unitary $n \times n$ matrices A and B . If A commutes with C , then A and B can be transformed simultaneously by a unitary similarity transformation so that A becomes $A_1 \dot{+} A_2$, and B becomes the product of a permutation matrix and a matrix $B_1 \dot{+} B_2$ where B_1 commutes with A_1 and B_2 has as characteristic polynomial $\prod (x^{r_i} - e_i)$, with $r_i > 1$ and $|e_i| = 1$.*

A special case arises when all the characteristic roots of A are different. In order to describe it, we introduce several definitions, some of which will only be needed later.

Definition 1. A generalized permutation matrix is a matrix which has zeros in each row (and column), apart from one entry which is an element of absolute value 1.

Definition 2. A block permutation matrix is a block matrix which has zero

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