

*Lower Bounds for Eigenvalues**

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1. Introduction. In this paper a method will be developed for finding lower bounds to the eigenvalues of semi-bounded self-adjoint operators. It is well known that upper bounds are usually comparatively easy to obtain by the Rayleigh-Ritz procedure. The problem of finding lower bounds is much more difficult. It was, however, known for a long time that rough lower bounds can be obtained either by changing the boundary conditions or by changing the operator. For a review of this classical subject see the Gibbs Lecture of H. WEYL [13].

Our method goes back to A. WEINSTEIN [12]. (See also the expositions in the recent book of S. H. GOULD [7] and the article by J. B. DIAZ [6].) In his procedure, A. WEINSTEIN used the problem giving rough lower bounds as a starting point or *base problem*. He then linked the base problem to the given problem by a sequence of *intermediate problems* which can be explicitly solved in terms of the base problem and which improve the lower bounds. The introduction of intermediate problems is the essential idea of the procedure which will be used in our paper.

In WEINSTEIN's work on plates, going back to 1935, the intermediate problems are obtained by changing the boundary conditions. By combining his lower bounds with the upper bounds given by the Rayleigh-Ritz method, he obtained an accuracy of up to 0.18 percent.

Later ARONSAJN, in a paper published in 1951 [1], emphasized that a base problem can be obtained by a change of the operator and outlined the construction of the intermediate problems. His theory deals with completely continuous operators. The solution of ARONSAJN's intermediate problems requires, as in WEINSTEIN's theory, the determination of the poles and the zeros of a meromorphic function given in its partial fractions representation. The question of the numerical calculation of the zeros of such a function is, as ARONSAJN points out, not an easy one.

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The investigation was initiated when the author was at the National Bureau of Standards, and a resumé of some of the results was published in the Proceedings of the National Academy of Sciences [3].