

Strong Comparison Theorems for Elliptic Equations of Second Order

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1. Introduction. Consider a harmonic function $u(x,y)$, continuous in some closed bounded region $\bar{D} (= D + \partial D)$. Then the values of u in the open region D cannot exceed its maximum on ∂D . Furthermore, if u assumes its maximum value M at any point in D , then u is identically equal to M in \bar{D} . These two assertions are known respectively as the "weak" and the "strong" maximum principle for harmonic functions.

If $u(x, y)$ and $v(x, y)$ are two harmonic functions, both continuous in \bar{D} , and $u \leq v$ in D , the harmonic function $w = u - v$ is continuous in \bar{D} and not greater than zero in the same closed region. The strong principle asserts that either $w < 0$ in D or $w \equiv 0$ in \bar{D} ; in other words, either $u < v$ in D or $u \equiv v$ in \bar{D} . This result we will call the strong comparison theorem for harmonic functions. It is equivalent to the strong maximum principle since the latter may be regarded as a special case of it. Thus if M is the maximum value attained by u in \bar{D} , $u \leq M$ in D and M is a continuous harmonic function in \bar{D} . We can conclude from the comparison theorem that either $u < M$ in D or $u \equiv M$ in \bar{D} , and this is the assertion of the strong maximum principle.

HOPF in [1] gave an elegant proof of the extension of the strong principle to continuous functions satisfying linear uniformly elliptic inequalities of the form

$$(1.1) \quad L(u) = \sum_{i,j} a_{ij} p_{ij} + \sum_i b_i p_i \geq 0$$

where a_{ij} and b_i are uniformly bounded functions of n independent variables x_i in an open region D and $p_i \equiv \partial u / \partial x_i$, $p_{ij} \equiv \partial^2 u / \partial x_i \partial x_j$. This generalised maximum principle also has a corresponding comparison theorem, and the fact that $L(M) = 0$ for any constant M can be used to prove the equivalence of these two results.

In this paper it will be shown that the comparison theorem is more general than the maximum principle in that it remains true not only for linear uniformly elliptic operators of the form $L(u) + c(x)u$, where $c(x)$ is uniformly bounded in D , but also for a general class of nonlinear elliptic operators.