

*Infinite Lie Groups and the Formal Aspects of Dynamical Systems*¹

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One of the first treatments of the problems of normal forms for differential equations near a singular point occurs in the thesis of POINCARÉ [9]. He shows that if two conditions on an analytic system of differential equations are satisfied, there exists an analytic change of coordinates such that in the new coordinate system the differential equations can be written in diagonal linear form. One of these conditions was to guarantee convergence and the other was of a formal nature. It was shown in [14] and [15] that in the real domain, if we allow C^∞ instead of analytic changes of coordinates, the linearization theorem is true under the assumption of the formal condition alone (that is, the theorem is true for C^∞ equations as well. On the other hand, it was shown in [14] that the formal condition is necessary for even a C^2 linearization. (For the problem of C^1 linearization *cf.* [5].) The true meaning of the formal condition was explained in [16] where it was shown that the set of normal forms is a maximal commutative subalgebra of the algebra of all local vector fields, thus analogous to a Cartan subalgebra of a finite dimensional algebra, and the formal condition is just the analogue of a regularity condition. It was also shown in [16] that analogous results hold for the group of all local volume preserving maps, and partial results were obtained for local Hamiltonian maps. In all of these proofs, the corresponding results for the formal situation were first proved. Thus it was shown in [16] that given any formal Hamiltonian vector field satisfying the right regularity condition there exists a formal Hamiltonian change of coordinates (canonical transformation) which brings the vector field to a normal form. Even this purely algebraic theorem with no convergence difficulties is far from trivial (*cf.* also [10]).

In this paper we discuss the algebraic structure of groups of formal power series transformations and algebras of formal power series vector fields in some detail. We will show how all the normal form theorems can be deduced from two quite simple lemmas in the theory of (finite dimensional) Lie algebras

¹ The substance of this paper was given in a lecture at the International Symposium on Differential Equations, Mexico City, Mexico, September 1959.