

On Imbedding Graphs in the Sphere

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Introduction. The problem of determining a necessary and sufficient condition for a graph to be imbeddable in a sphere is a classical problem with a classical solution. A graph G may be imbedded in a sphere if and only if it does not contain either of the two Kuratowski graphs as subgraphs. This solution has two practical difficulties. First, for very large graphs, say with several hundred branches, it is difficult to find Kuratowski subgraphs. Secondly, knowing that a graph can be imbedded is not the same as knowing an imbedding. It is with these more practical aspects of the problem that we concern ourselves in this paper.

Our purpose is to present a practical algorithm which will either imbed a given graph on the sphere or prove that this cannot be done. A rough outline of how to apply this algorithm in practice is given in the first section. The remaining sections are devoted to a mathematically precise formulation and proof of the algorithm.

It should be remarked that the proof of the validity of the algorithm does not use the theory of the Kuratowski graphs and can be read independently of that theory.

1. Outline of the Algorithm. We will assume that we have been given a model of our graph G constructed out of wires soldered together. It is our conviction that it is more practical to use this model as a sort of analogue computer than to program our procedure for a digital computer. We also assume that each wire has been tagged with the names of the two points or nodes which it joins.

In order to state our procedure, we will need certain simple concepts. The first is that of a "simple cycle." This is nothing more than a subgraph of G which can be reshaped to form a circle. Secondly we must consider the idea of a pair of points (x_1, y_1) on a simple cycle C "separating" the pair (x_2, y_2) on C . By this we mean that one of the two paths in C from x_1 to y_2 meets these points in the order x_1, x_2, y_1, y_2 .

Suppose now that C is a simple cycle in G . In general, if we break all solders on the points along C , the graph G will fall into pieces. Some of these are simply