Local Isometric Imbedding of Riemannian Manifolds with Indefinite Metrics

AVNER FRIEDMAN

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Introduction. In 1873 SCHLÄFLI [9] conjectured that a Riemannian manifold R_m with positive definite and analytic metric can locally be imbedded isometrically as a submanifold of a euclidean space E_n, provided $n \geqq \frac{1}{2}m\ (m+1)$. In 1926 JANET [5] described a method of proof which however (as he himself observed) was incomplete. In 1927 E. CARTAN [3] gave a proof based on his theory of Pfaffian forms. BURSTIN [1] in 1931 completed the proof of JANET and also extended it to the case in which the enveloping space is a given Riemannian space R_n with positive definite and analytic metric. Recently LEICHTWEISS [6] gave a new proof of BURSTIN's extension based much more substantially (than [1]) on the GAUSS-CODAZZI equations of differential geometry. His proof is more involved than that of BURSTIN. Theorems on global isometric imbeddings of R_m into E_n which are of class C^k $(k \geqq 3)$ provided the metric tensor of R_m is of class C^k have been established by NASH [8]. For compact R_m he assumes that $n \geqq \frac{1}{2}m\ (3m+11)$.

The first purpose of this paper is to extend the theorem of JANET-CARTAN-BURSTIN to Riemannian manifolds with indefinite metrics (such as the space of General Relativity). The metric tensors are still assumed to be analytic and non-degenerate (for semi-positive definite metric, see LENSE [7]). We prove the following theorem (Theorem 1): If the metric tensor g_{ab} of R_m has p positive and q negative eigenvalues $(p+q=m)$ and if the metric tensor g_{ij} of R_n has at least p positive and at least q negative eigenvalues (no restriction being made on the signature of the remaining eigenvalues), then there exist local isometric and analytic imbeddings of R_m into R_n, provided $n \geqq \frac{1}{2}m\ (m+1)$. Our proof, like that of BURSTIN, is based on the general outline of JANET but otherwise it is a new proof even in the case of positive definite metrics.

We also consider in this paper the question of imbedding a given submanifold R_m of R_n in a family $R_m(t)$ of isometric submanifolds. We prove (Theorem 2) the existence of such a family of analytic submanifolds which varies analytically

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