

On Quasi-Linear Elliptic Equations which Arise from Variational Problems

H. JENKINS*

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1. Introduction. Consider the non-parametric variational problem

$$\delta \iint F(u, v) \, dx \, dy = 0;$$

where $\varphi(x, y)$ is the extremal function, and $u = \varphi_x$, $v = \varphi_y$. Assume that $F_{uu}F_{vv} - F_{uv}^2 > 0$. The EULER equation is then the quasi-linear elliptic equation

$$(1) \quad F_{uu}\varphi_{xx} + 2F_{uv}\varphi_{xy} + F_{vv}\varphi_{yy} = 0.$$

The minimal surface equation arises if $F = (1 + u^2 + v^2)^{\frac{1}{2}}$, and has the property, first proven by S. BERNSTEIN [1], that any solution defined for all values of (x, y) , is necessarily a linear function. Generalizations of BERNSTEIN's theorem to wider classes of equations have been given by BERS [2], FINN [3], and FINN & GILBARG [4]. HEINZ [5] has obtained a new proof of BERNSTEIN's theorem as a consequence of a deeper result; namely, the existence of an *a priori* estimate of the curvature of a solution of the minimal surface equation.

A conservation law of equation (1) is a pair of functions $\Theta(u, v)$, $\Lambda(u, v)$ for which

$$\Theta(u, v)_x + \Lambda(u, v)_y = 0$$

in the domain of any solution $\varphi(x, y)$. Equation (1) always possesses three natural conservation laws, since, as is well known, the differentials

$$(2) \quad \begin{aligned} d\psi &= F_v \, dx - F_u \, dy, \\ d\xi &= F - vF_v \, dx + vF_u \, dy, \\ d\eta &= uF_v \, dx + F - uF_u \, dy \end{aligned}$$

are exact. In the case of the minimal surface equation, NITSCHKE [7] has used

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