

# *The Topology of Weak Convergence of Probability Measures*

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**1. Introduction.** Let  $X$  be a topological space,  $\mathfrak{F}$  a  $\sigma$ -field of subsets of  $X$ , and  $\Pi(X, \mathfrak{F})$  a family of probability measures on  $(X, \mathfrak{F})$ . Let  $C(X)$  be the family of all real, bounded, continuous functions on  $X$  and  $C^+(X)$  the subfamily of non-negative functions in  $C(X)$ . The notion of convergence most frequently used in probability theory is the following: A sequence  $\mu_n$  ( $\mu_n \in \Pi(X, \mathfrak{F})$ ) is said to converge weakly to  $\mu \in \Pi(X, \mathfrak{F})$  if and only if  $\int f d\mu_n \rightarrow \int f d\mu$  as  $n \rightarrow \infty$  for every  $f \in C(X)$ . An appropriate topology in  $\Pi(X, \mathfrak{F})$  in which sequential convergence is weak convergence is defined by the following subbasis  $\mathfrak{B}$  of neighborhoods in  $\Pi(X, \mathfrak{F})$ :  $\mathfrak{B}$  is the class of all sets of the form

$$W(\mu, f, \epsilon) = \left\{ \mu' \in \Pi : \left| \int f d\mu' - \int f d\mu \right| < \epsilon \right\}$$

where  $\epsilon > 0$  and  $f \in C^+(X)$ . This topology will be referred to as the  $W$ -topology (BLAU [1]) or the topology of weak convergence. Another topology which is occasionally considered for  $\Pi(X, \mathfrak{F})$  and which is helpful in the study of the  $W$ -topology is specified by the subbasis of neighborhoods of the form

$$N(\mu, G, \epsilon) = \{ \mu' \in \Pi(X, \mathfrak{F}) : \mu(G) < \mu'(G) + \epsilon \}$$

where  $\epsilon > 0$  and  $G$  is an open set in  $X$ . This topology was introduced by BLAU [1], and we shall refer to it as the Alexandrov topology or the  $A$ -topology since the sequential convergence in this topology was introduced by A. D. ALEXANDROV [2]. We shall refer to the neighborhoods of the two topologies as  $A$ -neighborhoods and  $W$ -neighborhoods respectively.

It is clear that interesting results in the study of  $\Pi(X, \mathfrak{F})$  can be obtained only if we impose further conditions on  $X$ ,  $\mathfrak{F}$ , and the family  $\Pi(X, \mathfrak{F})$ . The basic assumption on  $X$  made throughout this paper is that  $X$  is a Tikhonov space, *i.e.* that  $X$  is completely regular and  $T_1$ . Unless specifically defined otherwise, the family  $\mathfrak{F}$  will be assumed to be the class of all Borel subsets of  $X$ , *i.e.* the family generated by the open sets of  $X$ ;  $\Pi(X, \mathfrak{F})$  will be the family of all regular probability measures on  $\mathfrak{F}$ ; and  $\Pi_1(X, \mathfrak{F})$  the set of all regular finite measures on