

# *On Approximation with Respect to Certain Quadratic Norms*

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*Communicated by W. FELLER*

**1. Introduction.** Let  $f(x)$  be a fixed positive integrable function on the real line determining the HILBERT space  $\mathcal{H}_f$  of all BOREL functions  $\psi(x)$  for which

$$(1.1) \quad ||\psi||^2 = ||\psi||_f^2 = \int |\psi(x)|^2 f(x) dx$$

is finite, with the usual inner product  $(\psi_1, \psi_2) = (\psi_1, \psi_2)_f$  given by the integral of  $f\psi_1\psi_2$ . Here, and elsewhere throughout this paper unless the contrary is indicated, integrals are extended over the entire real line. Let  $E$  denote a fixed closed interval on the real axis and  $\varphi(x)$  the FOURIER transform of a complex-valued measure  $m$  on the real axis:

$$(1.2) \quad \varphi(x) = \int e^{ixt} dm(t).$$

Here the measure must be finite on the whole line. It will be clear from the context when this condition is required; otherwise measures occurring are finite on compact sets. Let  $S_b$  denote the support of a function, measure or distribution  $b$ , as the case may be. The considerations of the present note arose through the question of approximating  $\varphi(x)$  with FOURIER transforms of measures  $\mu$  such that  $S_\mu \subset E$  (cf. [1]). Let  $\mathfrak{M}_E$  denote this class of FOURIER transforms. Then there exists a unique extremal function in  $\mathcal{H}_f$  which realizes the best approximation,

$$(1.3) \quad \inf_{S_\mu \subset E} ||\varphi - \mathfrak{F}(\mu)||_f, \quad \mathfrak{F}(\mu)(x) = \int_E e^{ixt} d\mu(t).$$

In general the extremum (1.3) will not be realized by a function belonging to  $\mathfrak{M}_E$ . The reason for this is that if we restrict the norm (1.1) to  $\mathfrak{M}_E$  and transfer this norm to the corresponding measures  $\mu$ ,

$$(1.4) \quad ||\mu|| = \left\| \int_E e^{ixt} d\mu(t) \right\|_f,$$

then the property of completeness will be lacking; not every CAUCHY sequence in the sense of (1.4) will have a limit which is also a measure.