

Best Approximation Properties of the Spline Fit

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Introduction. The need in numerical analysis for a method of interpolation which produces first derivatives of high quality has led recently to the use of the "spline fit." This approximation, which is essentially the numerical analogue of the draftsman's spline, consists of joining the assigned points by sections of cubics, requiring that slopes and curvatures be continuous at the junction points. The marked superiority of the quality of the derivatives over those obtained from polynomials of interpolation is a consequence of the relaxation of the overall requirement of analyticity.

In the various papers which have been published on this technique, [1], [2], [3], [4], interest has centered on its minimum curvature property, on its role in best approximation in the sense of TCHEBYCHEV and on raising the requirements of smoothness on the composite curve.

It is the purpose of this paper to present some of the fundamental best-approximation properties which characterize the spline fit.

Description of the procedure. Suppose that the interval $a \leq x \leq b$ is divided into N intervals by $x_0 = a < x_1 < x_2 < \cdots < x_N = b$. Suppose that the function $F(x)$ is prescribed at these $N + 1$ values of x : $F(x_k) = F_k$, $k = 0, 1, \cdots, N$. Join the points (x_k, F_k) by cubics and require that first and second derivatives of the pieced curve be continuous at these points. For convenience let M_k , $k = 0, 1, 2, \cdots, N$, denote the second derivatives of the approximating function $y(x)$. Then

$$(1) \quad y'' = M_{k-1} \frac{x_k - x}{l_k} + M_k \frac{x - x_{k-1}}{l_k}, \quad x_{k-1} \leq x \leq x_k,$$

where $l_k = x_k - x_{k-1}$.

Integrating twice and requiring $y(x_k) = F_k$ gives, for $x_{k-1} \leq x \leq x_k$, the relations

$$(2) \quad y' = -\frac{M_{k-1}(x_k - x)^2}{2l_k} + \frac{M_k(x - x_{k-1})^2}{2l_k} + \frac{F_k - F_{k-1}}{l_k} - \frac{(M_k - M_{k-1})l_k}{6}$$