

# Stochastic Integrals and Non-linear Processes

E. J. McSHANE

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If the input to a device during time interval  $(t_1, t_2)$  is the current  $y(t_1) - y(t_2)$  of a function  $y$  and the output at a time  $t$  is a functional  $F_t(y)$  of the input, under sufficiently strong hypotheses on  $F$  and  $y$  the output has a representation

$$(0.1) \quad F_t(y) = f_0 + \int_{-\infty}^{\infty} f_1(\tau) dy(\tau) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_2(\tau_1, \tau_2) dy(\tau_1) dy(\tau_2) + \cdots$$

When  $y$  is a sample curve of a Brownian motion these integrals can be defined<sup>1</sup> as stochastic integrals, determined "with probability 1," even though  $y$  is (with probability 1) of unbounded variation on every interval. Such integrals have been investigated by several mathematicians,<sup>2,3</sup> and recently WIENER has emphasized their importance in applications.<sup>4</sup> However, all these investigations except one by ITÔ (*op. cit.*<sup>3b</sup>) have dealt only with Brownian motion processes. (This last reference I had overlooked until Professor ROSENBLATT mentioned it to me after this paper had been submitted for publication. The relation between ITÔ's integral and the present one is not very transparent. It is of course quite possible that the former could be used to obtain the results of this paper.) We shall now try to explain why the development of a theory of integration for other than Brownian motion processes is of some importance, and not merely an exercise in generalization.

Suppose that the terms in the right member of (0.1) have in some way been defined as random variables; the  $(1 + n)^{\text{th}}$  term, with integrand  $f_n$ , will be denoted by the symbol  $\tilde{f}_n$ . These random variables usually are involved in an application as follows: Two of them, say  $\tilde{f}_m$  and  $\tilde{f}_n$ , are defined and form the arguments of two functions  $\phi(\tilde{f}_m)$ ,  $\psi(\tilde{f}_n)$ ; the quantity of interest is the expected value of the product

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<sup>1</sup> WIENER, NORBERT, *The homogeneous chaos*, Am. Jour. Math., **60** (1939), pp. 897-936.

<sup>2</sup> KAKUTANI, SHIZUO, *Determination of the spectrum of the flow of Brownian motion*, Proc. Nat. Acad. Sci., **36** (1950), pp. 319-323.

<sup>3a</sup> ITÔ, KIYOSI, *Multiple Wiener integral*, Journal Math. Soc. Japan, **3** (1951), pp. 157-169.

<sup>3b</sup> ———, *Spectral type of the shift transformation of differential processes with stationary increments*, Trans. Amer. Math. Soc., **81** (1956), pp. 253-263.

<sup>4</sup> WIENER, NORBERT, *Non-linear Problems in Random Theory*, Technology Press of M. I. T. and John Wiley and Sons, N. Y., 1958.