

The Neumann Function for Simply Connected Regions

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1. Introduction. Classical formulas for the solution of the DIRICHLET and NEUMANN problems involve boundary integrals with kernels given in terms of GREEN's function or the NEUMANN function. These have been dealt with extensively in the literature, but for the most part only under certain restrictions as to smoothness of the boundary. However, in the case of simply connected plane regions one has access, *via* conformal mapping, to the wealth of detailed information available for the circle by FOURIER series methods. It is therefore natural to expect the formulas to remain valid for simply connected regions with fairly general boundaries.

The solution of the DIRICHLET problem from this point of view was first worked out by VERBLUNSKY [6] in 1951, the boundary being taken as a rectifiable JORDAN curve. We propose here to treat the NEUMANN problem from the same viewpoint, although the techniques turn out to be quite different. (We rely heavily on the weak convergence of certain families of mass distributions.) The solution of the generalized NEUMANN problem is thus arrived at in terms of the invariant NEUMANN function ([3], p. 492), and stability of the representation is established by proving certain convergence theorems. In addition, the classical relationships of the NEUMANN function with the DIRICHLET integral and the reproducing kernel are extended to the present setting. We show also that the reproducing kernel is actually applicable to a larger class of functions than the class Λ^2 for which it is generally defined.

2. Preliminaries. The circular neighborhood $\{\zeta: |\zeta - \zeta_0| < \rho\}$ will be denoted by $N_\rho(\zeta_0)$, its boundary by $C_\rho(\zeta_0)$, and its closure by $S_\rho(\zeta_0)$. We shall fix Ω as a plane region bounded by a rectifiable JORDAN curve Γ and take z_0 as a preferred point of Ω . There is then a unique function w mapping $N_1(0)$ conformally onto Ω with $w(0) = z_0$ and $w'(0) > 0$, and w can be extended to a homeomorphism of $S_1(0)$ onto $\bar{\Omega}$.

The classical NEUMANN function for $N_1(0)$ with pole at ζ is given by

$$N_\zeta^*(\tau) = -\log |\tau - \zeta| - \log |1 - \tau\bar{\zeta}|$$