

*A Characterization of the Catenoid*¹

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1. The present paper is concerned with doubly-connected minimal surfaces S in a slab $c_1 < z < c_2$ of (x, y, z) -space $(-\infty \leq c_1 < c_2 \leq \infty)$ (*i.e.* mappings into the slab from an annulus of the parameter plane) which have the property that their intersections S_c with every plane $z = c$, for $c_1 < c < c_2$, are JORDAN curves. There exist a few explicit examples, especially the catenoid (by which we mean here, as always, the complete catenoid) which appears as a particular case of the minimal surfaces determined by A. ENNEPER [1] and B. RIEMANN [7] (see Figure 1):

$$(1) \quad \begin{aligned} x(u, v) &= \alpha_0 + \alpha m(u) + u \cos v, \\ y(u, v) &= \beta_0 + \beta m(u) + u \sin v, \\ z(u, v) &= \gamma_0 + \int_{u_0}^u \frac{dw}{\Delta(w)} \end{aligned}$$

where $-\infty < u < \infty$, $0 \leq v < 2\pi$.

Here

$$m(u) = \int_{u_0}^u \frac{w^2 dw}{\Delta(w)}, \quad \Delta(w) = \sqrt{(\alpha^2 + \beta^2)w^4 + 2\gamma w^2 - 1},$$

and u_0 may be taken as $u_0 = \sqrt{(\delta - \gamma)/(\alpha^2 + \beta^2)}$ where $\delta = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$. If $\alpha = \beta = 0$ (the case of the catenoid) then γ must be positive and $u_0 = 1/\sqrt{2\gamma}$. The integrals can be expressed in terms of elliptic integrals of the first and second kind. The intersections S_c with planes $z = c$ are circles and, except for the catenoid, c_1 and c_2 are finite:

$$c_1 = \gamma_0 - \frac{1}{\sqrt{2\delta}} K(k), \quad c_2 = \gamma_0 + \frac{1}{\sqrt{2\delta}} K(k); \quad k = \sqrt{\frac{1}{2} \left(1 + \frac{\gamma}{\delta}\right)}.$$

$K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kind with modulus k .

Further examples for such doubly-connected minimal surfaces are secured

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