

*Kinematics of Homogeneous Turbulence**

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D. METRIC TRANSITIVITY

1. Introduction. In Parts A–C of this paper [1], we have given a measure-theoretic basis to the kinematics of homogeneous turbulence for incompressible velocity-fields filling all space at a given time t . In Part A, a random vector field (R.V.F.) was defined as a regular probability measure in the topological linear space Λ of all vector fields which are square integrable on compact sets. In Part B, it was proved that any covariance matrix $\Gamma_{jk}(\mathbf{x}, \mathbf{y})$ satisfying certain conditions was the covariance matrix of a unique *normal* R.V.F. In Part C, the spectral matrix measure $S_{jk}(\boldsymbol{\kappa})$ was defined. Under the usual assumptions of the theory of homogeneous isotropic turbulence, a one-one correspondence was established between admissible normal R.V.F. and energy distributions $\varepsilon(\boldsymbol{\kappa})$ in wave-number space.

Part D will be devoted to ergodic theory and metric transitivity. As is well known, (*cf.* [3]; also [10] pp. 52–55 and [11]) ergodic theorems are the most essential features of any statistical mechanics. Averages computed over a given sample are the only experimental “observables”. The first problem is to prove the existence of time-averages and of space-averages, at least for almost all samples. This problem has been solved by the *individual ergodic theorem* of GEORGE D. BIRKHOFF, originally stated for analytic manifolds; a proof for general probability measure spaces may be found in E. HOPF [3], p. 53. This theorem may be stated as follows.

Individual Ergodic Theorem. *Let $\{T_t\}$ be a one-parameter group of one-one measure-preserving transformations of a probability measure space $(\Omega, \mathfrak{F}, \mu)$ into itself. Then for any integrable function $f(\omega) \in L_1(\Omega)$, the average*

$$(1) \quad Ef(\omega) = \lim_{A \rightarrow \infty} \frac{1}{A} \int_0^A f(T_t \omega) dt$$

exists, for almost every sample $\omega \in \Omega$.

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