

# *A Remark Concerning a Lemma of Heinz on Harmonic Mappings*

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*Communicated by C. LOEWNER*

The following lemma is due to ERHARD HEINZ [1]:

Suppose that the transformation  $S: \begin{cases} X = X(\alpha, \beta) \\ Y = Y(\alpha, \beta) \end{cases}$  maps the unit disc  $\alpha^2 + \beta^2 \leq 1$  onto the unit disc  $X^2 + Y^2 \leq 1$  in a one-to-one and continuous manner. If  $X(\alpha, \beta)$  and  $Y(\alpha, \beta)$  are harmonic functions in  $\alpha^2 + \beta^2 < 1$  and  $X(0, 0) = Y(0, 0) = 0$ , then there exists an inequality,

$$\Phi(S) = [X_\alpha^2 + X_\beta^2 + Y_\alpha^2 + Y_\beta^2]_{\alpha=\beta=0} \geq \mu,$$

valid for all  $S$ .

A lower bound for  $\mu$ ,

$$\mu \geq 2 - \frac{8}{\pi} \sum_{k=2}^{\infty} \frac{1}{k^2} \approx 0.358,$$

found by HEINZ [1], was improved by JOHANNES C. C. NITSCHKE [2]:  $\mu \geq 0.64$ . In the present note it is shown that  $\mu > 0.869$ .

This estimate follows directly from HEINZ's proof of his lemma. Let  $\alpha = \rho \cos \vartheta$ ,  $\beta = \rho \sin \vartheta$  and  $Z(\alpha, \beta) = X(\alpha, \beta) + iY(\alpha, \beta)$ . Then

$$Z(\alpha, \beta) = \sum_{k=1}^{\infty} \rho^k (a_k \cos k\vartheta + b_k \sin k\vartheta)$$

with

$$\left. \begin{aligned} a_k &= \frac{1}{\pi} \int_0^{2\pi} Z(\cos t, \sin t) \cos kt \, dt \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} Z(\cos t, \sin t) \sin kt \, dt \end{aligned} \right\} \quad k = 1, 2, \dots$$

Since  $Z(0, 0) = 0$ ,

$$\int_0^{2\pi} Z(\cos t, \sin t) \, dt = 0.$$