A New Class of Probability Limit Theorems

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1. Introduction. The study of the asymptotic behavior of $S_n = \xi_1 + \ldots + \xi_n$, where ξ_i are independent, identically distributed, random variables with mean 0 and finite variance, is an important problem of probability theory whose solution has attained a considerable degree of perfection. The purpose of this paper is to begin to develop a theory of another rather extensive class of Markov processes along somewhat the same lines.

This vague statement can be clarified with an example: Suppose that $\{X_n\}$ is a *random walk* on the non-negative integers with the transition probabilities

$$(1.1) \qquad p_{i,i+1} = \frac{1}{2}\left[1 + \frac{\alpha}{i} + o\left(\frac{1}{i}\right)\right], \qquad p_{i,i-1} = 1 - p_{i,i+1}, \qquad i \neq 0,$$

$$p_{01} = 1 - p_{00} > 0, \qquad p_{ij} = 0 \quad \text{otherwise.}$$

We shall prove in § 2 that provided $\alpha > -\frac{1}{2}$, so that $\{X_i\}$ is null-recurrent or transient [6],

$$(1.2) \qquad \lim_{n\to\infty} \Pr\left(\frac{X_n}{\sqrt{n}} \leqq y\right) = \int_0^y \frac{2\xi^{2\alpha}e^{-\frac{1}{2}\xi^2}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha + \frac{1}{2})}\, d\xi, \qquad y \geqq 0.$$

This is our analogue of the central limit theorem giving the limit distribution of $S_n/\sqrt{n}$.

There are, of course, a large number of additional limit theorems for $\{S_n\}$. Many (but by no means all) of these are subsumed by the ERDOS-KAC-DONSKER *invariance principle* [3, 2] which says that, in a certain sense, the stochastic process $\{S_i/\sqrt{n},\ i = 0, 1, 2, \cdots\}$ converges to a Brownian motion (Wiener) process as $n \to \infty$; a large class of path functionals, such as max $(S_1/\sqrt{n}, \cdots, S_n/\sqrt{n})$, thus are shown to have limiting distributions. The greater part of the present paper is devoted to showing that a similar convergence occurs in the theory of processes like the random walks above. The limiting process is a Wiener process with reflecting barrier at the origin if $\alpha = 0$, but for other values

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