

Properties of Some Interpolation Spaces

J. L. LIONS

Communicated by A. ZYGMUND

1. Introduction. Let A_0 and A_1 be two Banach spaces, both contained in a vector topological space \mathcal{G} , the identity mapping from A_i into \mathcal{G} being continuous for $i = 0, 1$. If $\|a_i\|_{A_i}$ denotes the norm of $a_i \in A_i$, $i = 0, 1$, we set, for $a \in A_0 + A_1$,

$$\|a\|_{A_0+A_1} = \inf_{a_0+a_1=a} (\|a_0\|_{A_0} + \|a_1\|_{A_1}),$$

the inf being taken for all possible decompositions of $a = a_0 + a_1$, $a_i \in A_i$; $A_0 + A_1$ is a Banach space for $\|a\|_{A_0+A_1}$.

Let B_0, B_1, \mathcal{B} be a family of three spaces, with analogous properties.

We denote by $\mathcal{L}(A_i; B_i)$ the space of continuous linear mappings of A_i into B_i . We shall say that

$$(1.1) \quad \pi \in \mathcal{L}(A_0; B_0) \cap \mathcal{L}(A_1; B_1)$$

if, assuming $A_0 \cap A_1$ dense in A_0 and A_1 , π is a linear mapping of $A_0 \cap A_1$ into $B_0 \cap B_1$ which can be extended by continuity to an element of $\mathcal{L}(A_i; B_i)$, $i = 0, 1$. It is easily checked that if π satisfies (1.1), then

$$(1.2) \quad \pi \in \mathcal{L}(A_0 + A_1; B_0 + B_1).$$

Let A and B be two Banach spaces, $A \subset A_0 + A_1$ and $B \subset B_0 + B_1$. We say that $\{A, B\}$ is an interpolation couple if, for every π satisfying (1.1), one has $\pi \in \mathcal{L}(A; B)$.

A number of systematic constructions of interpolation couples are known: [10], [4], [11], [1], [2], [5], [8], [9], [12], [13], [14], and the problem of the comparison between the spaces obtained by these constructions arises and is useful in applications, cf. [15].

We begin to attack this problem by comparing two methods: (a) the trace method [10], [11], [12], [14];¹ (b) the method of CALDERON [2] and the author [13]. What one must know about these methods is recalled in sections 2 and 4.

¹ In [14] we use fractional derivatives. This introduces *new* spaces (in general). We shall compare these spaces with the one introduced by CALDERON and the author on another occasion.