

Modifications and Cobounding Manifolds, III.

ANDREW H. WALLACE

1. Introduction. The object of this paper is twofold. In the first place an alternative method is worked out for cancelling modifications, in the terminology of [4]. Although the result obtained is slightly weaker, the advantage here is simple connectedness is not required. All the manifolds in [4] were orientable and so, if not simply connected to begin with, could be made so by means of suitable modifications. This was always done in [4], and indeed essential use was made of simple connectedness in various places (for example in Lemma 8.1 and its consequences) although by some oversight the assumption of this condition was not always explicitly stated. The technique presented here can now be applied to non-orientable manifolds, which of course cannot be made simply connected by modifications shrinking 1-spheres carrying the generators of the fundamental group, because some of these will not be directly embedded.

The second and more important object is to obtain, using the new cancelling technique, an extension of SMALE's theorem on J -equivalence or h -cobordism, that is to say the relation between two manifolds M_1 and M_2 which together form the boundary of a manifold M such that M_1 and M_2 are both deformation retracts of M . SMALE's theorem says that this condition is equivalent to the homeomorphism of M_1 and M_2 if these manifolds are simply connected. In this paper it will be shown that this result still holds if simple connectedness is replaced by the weaker invariance condition described in §§4, 5 below.

2. Rearrangement of modifications. The two theorems of this section prepare the way for the cancelling theorem of the next section.

Theorem 1. *Let M_1 be transformed into M_2 by a sequence of modification of types $(s, n - s - 1)$, $s \geq r > 1$, $n \geq r + 2$, and let the trace of these modifications be M . Let α be in the kernel of the inclusion $\pi_r(M_1) \rightarrow \pi_r(M)$. Then the sequence of modifications can be arranged, without changing M , so that one of the $(r, n - r - 1)$ modifications shrinks a sphere carrying α .*

Proof. Let $\beta_1, \beta_2, \dots, \beta_k \in \pi_r(M_1)$ be carried by the spheres to be shrunk in the $(r, n - r - 1)$ modifications. Then, by the argument of Lemma 8.1 of [4], α is a linear combination of the β_i and their transforms under the elements of $\pi_1(M_1)$. Pick one of the β_i (this is an arbitrary choice), say β_1 , and write