

# Some Markovian Optimization Problems

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In this paper we shall prove the existence and uniqueness of a solution in a bounded cylindrical domain for nonlinear parabolic partial differential equations of the form

$$(1) \quad \frac{\partial u}{\partial s} + \sum_{i,j=1}^n a_{ij}(s, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \max_{u \in K} \Phi(y; s, x, u, \nabla u) = 0,$$
$$\Phi(y; s, x, u, \nabla u) = F(s, x, y) + \sum_{i=1}^n G_i(s, x, y) \frac{\partial u}{\partial x_i} + H(s, x, y)u,$$

together with given mixed boundary-terminal data. For  $H = 0$ , this is the basic equation for Markovian optimization problems of the type which we are going to consider. It also has other applications in the theory of nonlinear parabolic equations (see §6).

The solution  $u^*(s, x)$  is obtained as the limit of a monotone sequence  $u^{(r)}(s, x)$  of solutions of certain linear parabolic equations. This sequence is defined by a procedure suggested by R. Bellman [1, p. 171] for a related deterministic optimization problem. The success of the method depends primarily on certain *a priori* estimates due to A. Friedman [8, 9] and to E. Gagliardo [10, 11].

**1. Background.** Many problems of optimum control fall into the following class. There is given a system with  $n$  state variables and  $m$  control variables

$$\xi = (\xi_1, \dots, \xi_n), \quad y = (y_1, \dots, y_m),$$

governed by a system of ordinary differential equations

$$(2) \quad d\xi_i = G_i(t, \xi, y) dt, \quad i = 1, \dots, n,$$

together with given initial data for  $\xi$ . The problem is to choose  $y$  as a function  $y(t)$  of time  $t$  in order to optimize some specified quantity. There is an extensive literature on such problems (see, for example, [1], [3], [17]).

It is also of considerable interest to study the case when random effects (*e.g.* noise) enter into the system. There then arises an important distinction

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