

Maximal Vector Spaces of Light Interior Functions

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1. **Introduction.** If f and g are light, interior functions defined in a domain \mathfrak{D} in the complex plane, the function $\alpha f + \beta g$, (α, β real numbers) need not be either light or interior. If \mathfrak{W} is a real linear vector space of functions which are light, interior, and orientation-preserving in \mathfrak{D} , one may inquire if there is some property which characterizes elements of \mathfrak{W} . One class of light interior functions which has proved of interest is the class of pseudo-regular functions. In 1954, Titus & McLaughlin showed that if \mathfrak{W} is a real linear vector space of pseudo-regular functions in \mathfrak{D} which contains two analytic functions f and g such that $\text{Im} (f'\bar{g}') \neq 0$ in \mathfrak{D} , then \mathfrak{W} contains only analytic functions. In the same year, Golomb proved that if \mathfrak{W} contains two functions $f = u + iv$ and $g = p + iq$ such that $v_x q_u - v_u q_x \neq 0$ in \mathfrak{D} , then \mathfrak{W} contains only solutions of a uniquely determined elliptic system of first-order partial differential equations.

It follows from Golomb's theorem that if \mathfrak{W} consists of the solutions of an elliptic system \mathcal{L} in a domain \mathfrak{D} , then \mathfrak{W} is maximal in the set of functions that are pseudo-regular in \mathfrak{D} . One is led to wonder if the property of *ellipticity* of \mathcal{L} is essential to preserve the topological and algebraic properties of elements of \mathfrak{W} or whether it is enough that \mathcal{L} be elliptic in \mathfrak{D} except for some "sufficiently thin" set Z on which \mathcal{L} may be parabolic. The latter proposition is proved in Theorem 3.1 under the additional hypothesis that the coefficients of \mathcal{L} be Hölder-continuous in \mathfrak{D} .

Two classes of elliptic systems which are of particular interest are the Bers systems and the Beltrami systems. In 1938, Morrey proved that corresponding to each Jordan domain \mathfrak{D} and each Beltrami system \mathcal{L} in \mathfrak{D} , there exists a homeomorphism h of \mathfrak{D} onto itself such that the class of functions that are solutions of \mathcal{L} is the class of analytic functions of h . In the same year, Kakutani proved that if \mathfrak{W} is a collection of pseudo-regular functions in \mathfrak{D} , a necessary and sufficient condition for \mathfrak{W} to be a ring is that all elements of \mathfrak{W} are analytic functions of a fixed pseudo-regular function. In Theorem 3.3 it is shown that if \mathfrak{W} is the set of solutions of a system \mathcal{L} of first-order partial differential equations which is elliptic except at most on a compact, zero-dimensional set Z , a necessary and sufficient condition for \mathfrak{W} to be a maximal algebra of light, interior, orienta-