

# *Modifications and Cobounding Manifolds, IV*

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**Introduction.** This paper is a sequel to [4], [5], [6] and the notations and definitions of these papers will be used here.

It has already been shown [6] that if cobounding manifolds  $M_1$  and  $M_2$  contain basic complexes, then it is possible to transform one into the other by a specially simple sequence of modifications. The object here is to try to simplify this sequence still further in the even dimensional case. In the present paper  $M_1$  and  $M_2$  will be joined together, thus forming a bounding manifold, or, what is the same thing, the case  $M_2 =$  a sphere will be considered. Then  $M_1$  bounds a manifold  $M$ , and the crucial point in the reduction is the killing of the homology of  $M$  modulo  $M_1$ . This involves generalizing the results of Wall [4] to the relative case, and this extension occupies most of the present paper. The final result, giving a simplified sequence of modifications transforming an even dimensional bounding manifold to a sphere, appears as Theorem 10.2.

**1. A manifold with boundary and basic complex.** Let  $M$  be a differentiable manifold of dimension  $2m + 1$  with boundary  $M_1$ . Let  $M$  contain a basic complex  $L$ , satisfying the conditions of [6], §9, up to dimension  $m - 1$ . Also let  $M_1$  contain a basic complex  $L_1$ , and suppose that this has been arranged, as in [6], Theorem 9.7, to satisfy the basic complex conditions up to dimension  $m - 1$ .

It is desirable at this point to ensure that  $H_m(M)$  has a free generator carried by a skew embedded  $m$ -sphere in  $M_1$ . If this condition is not already satisfied, the following adjustment must be made. Apply to  $M_1$  a modification  $\phi$  of type  $(m - 1, m)$ , shrinking the boundary of an  $m$ -cell which does not meet  $L_1$ , and carry out the modification in such a way that this cell is closed up to form an  $m$ -sphere whose normal bundle in  $\phi(M_1)$  has characteristic class mapping on 1 under the inclusion  $\pi_{m-1}(R_m) \rightarrow \pi_{m-1}(R_{m+1})$  when the latter group is  $Z$  or  $Z_2$ . Of course when  $\pi_{m-1}(R_{m+1}) = 0$  the question of introducing this condition does not arise. Denote by  $M'_1$  the result of applying the modification  $\phi$ , and let  $M'$  be the union of its trace with  $M$ , the common subset  $M_1$  being identified in the obvious way. There are two  $m$ -spheres in  $M'_1$  associated with  $\phi$ , one obtained by closing up an  $m$ -cell as just described, the other being the sphere introduced by the modification in the sense of [6], §3. The latter bounds a cell in  $M'$ , while the former carries a cycle in  $M'$  which is not homologous to zero. Let  $\alpha_0$  be the corresponding element of  $H_m(M')$ .

Comparing the homology of  $M'$  with that of  $M$ , it is clear that  $H_i(M) = H_i(M')$  for  $i < m$ , and  $H_m(M') = H_m(M) \oplus \{\alpha_0\}$ , where  $\oplus$  denotes direct summation