

# *Topologies Generated by Outer Measures*

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**1. Introduction.** In [2] and [3], it was shown that  $n$ -dimensional Lebesgue measure induces a topology on Euclidean  $n$ -space in the following way: a set  $S$  is said to be open if it is measurable and if the metric density of  $S$  exists and is equal to one at every point of  $S$ . An interesting property of this topology is that the class of continuous function on  $E_n$  with this topology is precisely the class of approximately continuous functions. In [2], it was established that this topology is the coarsest one for which the approximately continuous functions are continuous, by showing that it is completely regular.

In this paper, it is shown that a regular outer Caratheodory measure which is finite on compact sets also induces a topology on Euclidean  $n$ -space in a similar manner. As in [2], we find that this topology is not normal but that it is completely regular. Complete regularity follows from the main theorem which is an analogue of the Lusin-Menchoff Theorem, [8]. Other questions concerning this topology are also investigated and it is found that, in general, two different measures induce distinct topologies. However, the topologies have most topological properties in common except connectedness.

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**2. Definitions and conventions.** We will let  $E_n$  denote Euclidean  $n$ -space and  $L_n$   $n$ -dimensional Lebesgue measure over  $E_n$ . An *interval* will always mean an *oriented interval*, and if  $p \in E_n$ ,  $r > 0$ ,  $Q_p^r$  will denote the *open cube* with  $p$  as center and  $r$  as its diameter. If  $A \subset E_n$ ,  $\delta(A)$  will mean the diameter of  $A$ ,  $\bar{A}$  will be its topological boundary,  $\tilde{A}$  its complement,  $A^\circ$  its interior, and  $\bar{A}$  its closure. Throughout this paper, an *outer measure*,  $\varphi$ , will mean a Caratheodory outer measure which is finite on compact sets and has the property that every set is contained in a  $G_\delta$  set of equal outer measure. In order to avoid technical difficulties, which could be easily overcome, we will also assume that  $\varphi(Q_p^r) > 0$

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