

*Representation Theorems for Positive Functions*¹

SAMUEL KARLIN

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1. Introduction. Let $u_0(t), \dots, u_n(t)$ represent continuous functions on a finite interval $[a, b]$. These functions are called a Tchebycheffian system or T-system provided that all the $(n + 1)^{\text{st}}$ order determinants

$$(1) \quad U \begin{bmatrix} t_0, & \dots, & t_n \\ 0, & \dots, & n \end{bmatrix} = \begin{vmatrix} u_0(t_0) & u_1(t_0) & \dots & u_n(t_0) \\ u_0(t_1) & u_1(t_1) & \dots & u_n(t_1) \\ \vdots & & & \vdots \\ u_0(t_n) & u_1(t_n) & \dots & u_n(t_n) \end{vmatrix}$$

are of one strict sign, where t_0, t_1, \dots, t_n satisfy

$$a \leq t_0 < t_1 < \dots < t_n \leq b.$$

Without restricting generality (multiply $u_n(t)$ suitably by $+1$ or -1), we may assume that the determinants in (1) are positive.

Systems of functions of this type occur frequently in various branches of mathematics and probability theory. For example, if $u_i(t) = t^i, i = 0, \dots, n$, then (1) reduces to the familiar Vandermonde determinant. As a consequence of Gantmacher & Krein [5] it turns out that for Sturm-Liouville eigenvalue problems with discrete positive spectrum the first $n + 1$ eigenfunctions $\varphi_0, \varphi_1, \dots, \varphi_n$ form a T-system. More generally the first $n + 1$ eigenfunctions associated with an integral transformation

$$T\varphi = \int_a^b K(x, y)\varphi(y)d\sigma(y) \quad d\sigma \geq 0,$$

where $[a, b]$ is finite and K has an iterate which is strictly totally positive, *i.e.*, satisfies certain determinantal inequalities, constitute a T-system [4].

T-systems play a role in the interpolation problem, the moment problem, the study of oscillation properties of polynomials and in other branches of analysis.

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