

*The Invariance of the Index in the Metric Space of Closed Operators**

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In this paper we attempt to give a systematic treatment of the properties of semi-Fredholm operators on a Hilbert space. Throughout this article, a semi-Fredholm operator will be a closed operator—not necessarily bounded—with a dense domain and a closed range, for which an index (*cf.* previous definitions of F. Noether [14] and F. V. Atkinson [1]) can be defined.

§1 is devoted to the definition and characterization of those operators.

§2 is concerned with the definition and algebraic properties of the index of semi-Fredholm operators. The main result of this paragraph was first established by Atkinson [1] for the special case of bounded operators, and then extended to unbounded operators (with finite indices) by Gokberg and Krein [8].

In §3, using the notion of the graph of a linear operator we introduce a metric d on the set of all closed operators on a given Hilbert space \mathfrak{H} , as has been previously done by Newburgh [13]. This is achieved by defining first the distance between two closed subspaces of a Hilbert space and then taking the distance between two closed operators to be the distance between their graphs. An equivalent metric is obtained by using the notion of gap between closed subspaces—introduced by Krein & Milmann [15]. Finally a third metric equivalent to the first two is defined in term of the operator norm of $R_A = (1 + A^*A)^{-1}$, R_{A^*} , AR_A .

On the subset of all bounded closed operators on \mathfrak{H} , there is a simple relation between the topology induced by $d(A, B)$ and that induced by the metric $s(A, B) = \|A - B\|$.

In §4 & 5 an interesting relation between the distance of two semi-Fredholm operators and the distance of their null spaces is established, which together with a lemma proved by Nagy [16] leads to a relation similar to Atkinson's first stability theorem [1, 2] and its generalizations by Krein & Krasnosel'skii [7] and Nagy [9]. The set \mathfrak{F} of all semi-Fredholm operators on \mathfrak{H} is shown to be open in the set of all closed operators on \mathfrak{H} with respect to the topology introduced in §3. Finally it is proved that there exists around each element

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