

Best Approximation Properties of Spline Functions of Odd Degree

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Introduction. In [4], interpolation by cubic spline functions is discussed, and some best approximation properties of the cubic spline fit are described. This note extends the results of [4], in a somewhat modified form, to spline functions of odd degree $m = 2k - 1$, $k \geq 2$.

Definition. A spline function of degree m with joints $\xi_1 < \xi_2 < \dots < \xi_n$ is defined as a function $F(x)$ with the following two properties ([2], p. 67):

- a. In each of the intervals $(-\infty, \xi_1)$, $[\xi_1, \xi_2)$, \dots , $[\xi_n, \infty)$, $F(x)$ is a polynomial of degree m ;
- b. $F(x)$ has continuous derivatives through the $(m - 1)^{\text{st}}$, or, for short, $F(x) \in C^{m-1}$.

The class of functions $F(x)$ with these properties will be denoted by $S_m(\xi_1, \dots, \xi_n)$.

The following lemma establishes the existence and uniqueness of a spline function of degree $(2k - 1)$ with $(n - 1)$ joints which coincides with a given function $f(x)$ at $(n + 1)$ prescribed points. The lemma is a consequence of Theorem 2 in [3], p. 258.

Lemma 1. Let $f(x)$ be any function of class $C^k[a, b]$. For each choice of $n + 1$ abscissae x_i , $a = x_0 < x_1 < \dots < x_n = b$, there exists exactly one spline function in $S_{2k-1}(x_1, \dots, x_{n-1})$, denoted by $\bar{s}(x)$, such that

- (1) $\bar{s}(x_i) = f(x_i), \quad i = 0, \dots, n,$
- (2) $\bar{s}^{(k+i)}(x_i) = 0, \quad i = 0, n; \quad j = 0, \dots, k - 2,$

where $\bar{s}^{(m)}(x)$ denotes the m^{th} derivative of $\bar{s}(x)$.

It has been known for some time (cf., e.g. [2], p. 67) that in the case $k = 2$ of cubic splines the interpolating function $\bar{s}(x)$ minimizes $\int_a^b [u''(x)]^2 dx$ among all functions $u(x) \in C^2$ which coincide with $f(x)$ at the points x_i , $i = 0, \dots, n$. The