

On Forms of Degree n Permitting Composition¹

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Any finite-dimensional separable alternative algebra A over a field F is a direct sum $A = A_1 \oplus \cdots \oplus A_r$ of simple ideals A_i , where the center Z_i of A_i is a separable extension of F of degree d_i . The generic norm $n_i(x_i)$ for A_i ([6], p. 180) is a (homogeneous) form of degree $m_i d_i$ over F where m_i is the degree of A_i as a central simple algebra over Z_i (in case A_i is not associative, A_i is a Cayley algebra over Z_i and $m_i = 2$). Let f_1, \dots, f_r be arbitrary positive integers, and $x = x_1 + \cdots + x_r$, x_i in A_i . Then

$$(1) \quad N(x) = [n_1(x_1)]^{f_1} \cdots [n_r(x_r)]^{f_r}$$

is a form of degree

$$(2) \quad n = \sum_{i=1}^r f_i m_i d_i$$

on A which permits composition:

$$(3) \quad N(xy) = N(x)N(y) \quad \text{for all } x, y \text{ in } A.$$

If F is of characteristic 0 or $p > n$, $N(x)$ is nondegenerate.

It is conjectured in [9] that, conversely, any (possibly infinite-dimensional) algebra A with 1 over F of characteristic 0 or $p > n$, on which a nondegenerate form $N(x)$ of degree n permitting composition is defined, is a finite-dimensional separable alternative algebra with $N(x)$ given by (1) and (2). This is a classical result ([1], [5], [7]) for quadratic forms, the dimension of A being 1, 2, 4 or 8. For finite-dimensional A the theorem is known for cubic forms ([9]), the dimension of A being 1, 2, 3, 5 or 9. In this paper we prove for arbitrary n , without assuming finite-dimensionality, that A is alternative (Theorem 2). We prove also that the elements of A satisfy well-determined equations of degree n .

Assuming that A is finite-dimensional, we prove that A is a separable (alter-

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