

*Characterizing Normal Law and a Nonlinear Integral Equation*¹

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1. Introduction. Let X_1, \dots, X_n be independent random variables (r.v.'s) with common distribution function (d.f.) $F(x)$. If two different functions of these r.v.'s (usually one linear and the other non-linear) are given to be statistically independent, then in a number of cases the d.f. $F(x)$ was characterized completely. For instance if these functions are the mean and the variance, then $F(x)$ was normal. (See Lukacs [6].) A general method is to translate the conditions of independence, usually through Fourier transforms, into a differential equation whose solution then yields the desired result. Certain other related methods were also discussed in [6]. However, the use of integral equations in these problems did not seem to have been considered in the past. The purpose of this paper is to characterize the normal distribution through a *nonlinear* integral equation.

The problem considered in this paper is to characterize the d.f. $F(x)$, given that the mean and a "tube statistic" (to be defined below), formed of the above r.v.'s, are independently distributed. The so-called tube statistics include "sample range", "sample central absolute moments", etc. In the present set-up the problem leads naturally to a non linear integral equation. Under some mild restrictions on the class of d.f.'s $F(x)$, such as continuity and behavior in the tails, the solution of the integral equation was found to be normal, thus giving the characterization of $F(x)$. On the other hand, if $F(x)$ is three times differentiable, the problem was solved in a different way (reducing to a differential equation) by Paskevich [7]. The solution of the integral equation presented here may itself be of independent interest since very little seems to be known about such equations in the literature.

Before proceeding to the next section, it is of interest to note a consequence

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