

Wiener-Hopf Difference Equations

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1. The theory of mixed problems for linear hyperbolic equations appears to be surprisingly undeveloped in comparison with that of pure initial-value problems. This is certainly true of their difference analogues, and for very much the same reason: the usual Fourier transform methods fail in the presence of boundaries. Without boundaries, we are able in principle to decide on the convergence of most proposed difference methods, and one of the remaining problems is simply to sort out those which are useful in practice. For mixed problems, although the convergence of several important methods has been proved by means of energy identities [1-3], a general theory is completely lacking.

We shall make a beginning by developing such a theory in the simplest situation, that is for a scalar equation and one plane boundary. This is a problem which still yields to a Fourier transformation, or more precisely to two, after a Wiener-Hopf factorization. Our conclusion is that the familiar von Neumann condition, supplemented by an "index condition" on the implicit part of the difference operator, is necessary and sufficient for convergence. To achieve a result of this kind for equations with variable coefficients, when the operators involved are no longer purely convolutions over a half space, we have to extend somewhat the factorization theory on which the Wiener-Hopf technique rests. For example, we find a necessary and sufficient condition for the factorization into invertible triangular operators of $I + K$, where K is completely continuous on l_1 . We intend to take up elsewhere the continuous analogue, in which a Fredholm operator becomes the product of Volterra operators, as well as mixed problems with two boundaries.

Finally, we examine the order of accuracy which can be attained by a certain class of convergent methods. We already know this order to be unbounded for initial-value problems [4], and bounded by *two* for explicit methods in the present case [5]. Our new result actually lies entirely within the framework of complex variables. The problem is to approximate z^μ , $\mu < 0$, by a rational function $r(z)$

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