

Deformations of Linear Connections and Riemannian Manifolds

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Introduction. The aim of this paper is to develop a theory of deformations of linear connections and Riemannian metrics on a differentiable manifold. The results obtained are analogous to those in the theory of deformations of complex structures developed by Kodaira-Spencer in [4].

We first give a criterion for local triviality (Theorem 4.1) of a family of deformations of complete linear connections (resp. complete Riemannian metrics). For deformations of complete linear connections (resp. complete Riemannian metrics) on a manifold with a finitely presentable fundamental group, under an assumption on the connection (resp. metric) which we call regularity, we prove stronger results (Theorems 8.1, 9.1 and 9.2). The hypothesis of regularity, which assures us that the sheaf of germs of Killing vector-fields is a local coefficient system, is sufficiently general to include the case of analytic or locally homogeneous linear connections (and Riemannian metrics). Our results apply in particular to the case of deformations of compact Riemannian manifolds with constant curvature treated by E. Calabi in [1]. (*cf.* Remark in §11).

Our methods of proof make no use of harmonic forms, though the theory of harmonic forms with coefficients in a local system of vector spaces can be applied to treat the case of deformations of regular linear connections (and regular Riemannian metrics) on a compact manifold. Instead, we use cohomology theory of groups which enables us to treat a wider class of manifolds, *viz.* manifolds with finitely presentable fundamental groups.

The following is a consequence of our results: Let G be a simply connected Lie group and U an open neighbourhood of the origin 0 in n -dimensional Euclidean space R^n . Let Γ be a discrete group and ρ_t , $t \in U$ be a differentiable family of isomorphisms of Γ in G such that $\rho_0(\Gamma)$ is a discrete subgroup of G with compact quotient. If further $H^1(\rho_0(\Gamma), \mathfrak{G}) = 0$ where \mathfrak{G} , the Lie algebra of G , is considered as a $\rho_0(\Gamma)$ -module for the adjoint representation of $\rho_0(\Gamma)$ in \mathfrak{G} , then, $\rho_t(\Gamma)$ and $\rho_0(\Gamma)$ are inner conjugates for t sufficiently near to 0 .

In §12, we show that some results of A. Weil (§§7–10 of [13]) lead to the