

# *A Boundary Value Problem Arising in the Theory of Univalent Functions*

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## INTRODUCTION

In the past fifty years a great deal of attention has been given to the study of inequalities satisfied by the coefficients of univalent functions. Consider the class  $S$  of holomorphic functions  $w = f(z)$  with the Taylor series

$$f(z) = z + a_1z^2 + a_3z^3 + a_4z^4 + \dots$$

which are univalent in the unit disc. These map the unit disc conformally onto domains  $D$ . In 1916 Bieberbach [1] showed that, for all functions in  $S$ ,  $|a_2| \leq 2$ , with equality only for the Koebe function

$$\frac{z}{(1 - e^{i\theta}z)^2} = z + 2e^{i\theta}z^2 + 3e^{2i\theta}z^3 + 4e^{3i\theta}z^4 + \dots$$

This led him to conjecture  $|a_n| \leq n$ , and again with equality only for the Koebe function. Since then, in 1923 Loewner [11] proved  $|a_3| \leq 3$ , while in 1955 Garabedian—Schiffer [8] proved that  $|a_4| \leq 4$ .

Other conjectures arose for related classes of functions. We shall mention two of these since they have a bearing on our attitude. For the subset of  $S$  of odd univalent functions with Taylor coefficients  $c_{2n+1}$ , it was conjectured that

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