

# *Padé Approximants as Limits of Rational Functions of Best Approximation\**

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We shall call a rational function of type  $(n, \nu)$  provided it can be written in the form

$$\frac{s_0 + s_1 z + \cdots + s_n z^n}{t_0 + t_1 z + \cdots + t_\nu z^\nu}, \quad \sum |t_k| \neq 0.$$

In a thesis [1] that has become classical, Padé studied for each pair  $(n, \nu)$  the rational function  $P_{n\nu}(z)$  of type  $(n, \nu)$  which has contact of highest order at the origin with a given analytic function

$$(1) \quad f(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots, \quad a_0 \neq 0.$$

Later Montessus de Ballore [2] studied the convergence of the  $P_{n\nu}(z)$  and showed under suitable mild conditions that if  $f(z)$  is a function of  $z$  meromorphic with precisely  $\nu$  poles in the disk  $D_0: |z| < \rho$  ( $\leq \infty$ ) and if  $D$  denotes that disk with the  $\nu$  poles deleted, then the sequence  $P_{n\nu}(z)$  of Padé functions approaches ( $n \rightarrow \infty$ )  $f(z)$  uniformly on any closed bounded set in  $D$ , and the finite poles of  $P_{n\nu}(z)$  approach respectively these  $\nu$  poles of  $f(z)$ . With this same condition on  $f(z)$ , the present writer has shown [3] (as a special case of a much more general result) that if  $f(z)$  is also analytic in the closed disk  $\delta: |z| \leq \epsilon < \rho$ ,  $\epsilon > 0$ , then the rational functions  $R_{n\nu}(\epsilon, z)$  of type  $(n, \nu)$  of best Tchebycheff approximation to  $f(z)$  on  $\delta$  converge uniformly ( $n \rightarrow \infty$ ) to  $f(z)$  on any closed bounded set in  $D$  and the finite poles of  $R_{n\nu}(z)$  approach respectively these  $\nu$  poles of  $f(z)$ ; the method of proof is by study of degree of convergence in  $\delta$  and  $D$  of the sequence  $R_{n\nu}(\epsilon, z)$ .

The object of the present note is to show that there is more than an analogy between these two sets of rational functions, by showing (i) that under suitable conditions the rational functions  $R_{n\nu}(\epsilon, z)$  for fixed  $n$  and  $\nu$  approach ( $\epsilon \rightarrow 0$ ) the Padé function  $P_{n\nu}(z)$ , and (ii) that for fixed  $\epsilon$  the geometric degrees of convergence ( $n \rightarrow \infty$ ) to  $f(z)$  of the sequences  $P_{n\nu}(z)$  and  $R_{n\nu}(\epsilon, z)$  in that part of  $D$  exterior to  $\delta$  are the same. In this sense the Padé functions are limit cases

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