

On Some Classes of Continuous Functions with Convergent Fourier Series

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Introduction. In this work we will study real valued continuous functions which are periodic of period 2π . For reasons of convenience these functions will be assumed, unless otherwise stated, normalized to have minimum zero and maximum one. Given such a function $f(x)$ we set

$$E_y = \{x: f(x) > y\}.$$

In view of the periodicity assumptions we can visualize E_y as a subset of the unit circle. Since E_y is open, it will be a union of open intervals; let $N(E_y)$ be the number of such intervals (finite or infinite) counted on the unit circle.

S. Banach [1] considered, instead of $N(E_y)$, the function $n(y_0)$ which counts the number of intersections of the graph of $f(x)$ with the line $y = y_0$. However, for any given $f(x)$, the relation

$$n(y) = 2N(E_y)$$

holds for almost every y . The function $N(E_y)$ could well be infinite on a set of positive measure, nevertheless from a result of N. Bary [2] it follows that every continuous function is the sum of three continuous functions each having an almost everywhere finite $N(E_y)$.

Our main results are concerned with the classes $\mathcal{C}_{\log N}$ and $\mathcal{C}_{(\log N)^2}$ of functions satisfying respectively the two conditions

$$(I.1) \quad \int_0^1 \log N(E_y) dy < \infty \quad \text{and} \quad (I.2) \quad \int_0^1 [\log N(E_y)]^{1/2} dy < \infty.$$

Let $S_n(x, f)$ denote, as customary, the n^{th} partial sum of the Fourier series of $f(x)$. These results can be stated as follows.

Theorem I.1. *If $f(x)$ is in $\mathcal{C}_{\log N}$, then*

$$S_n(x, f) \rightarrow f(x)$$

uniformly in $[-\pi, \pi]$.