

# *Normal Extensions of Unbounded Formally Normal Operators*<sup>1</sup>

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**1. Introduction.** If  $A$  is an operator in a Hilbert space  $\mathfrak{S}$ , we will denote by  $\mathfrak{D}(A)$  its domain, and by  $\mathfrak{N}(A)$  its null space. A *formally normal operator*  $N$  in  $\mathfrak{S}$  is a densely defined closed operator such that  $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$ , and  $\|Nf\| = \|N^*f\|$  for all  $f \in \mathfrak{D}(N)$ . We shall only be concerned with *unbounded* formally normal operators in this paper. A *normal operator*  $N$  in  $\mathfrak{S}$  is a formally normal operator satisfying  $\mathfrak{D}(N) = \mathfrak{D}(N^*)$ . In an earlier work [1] one of the authors gave necessary and sufficient conditions in order that a formally normal operator  $N$  have a normal extension  $N_1$  in the Hilbert space  $\mathfrak{S}$ . These conditions concern the space  $\mathfrak{N}(I + N^*\bar{N}^*)$ , where  $I$  is the identity operator and  $\bar{N}$  is the restriction of  $N^*$  to  $\mathfrak{D}(N)$ . The results were obtained by applying the von Neumann extension theory for symmetric operators. Previous work on this problem was done by Y. Kilipi [3], [4] and R. H. Davis [2].

In §2 we give a more direct and geometrical derivation of the above-mentioned results by exploiting the relationship between the graphs of  $N$  and an extension  $N_1$ . We then show how the von Neumann result, characterizing when a symmetric operator can be extended to a self-adjoint one, appears as a consequence.

In §3 we consider the problem of extending a maximal formally normal operator  $N$  in a Hilbert space  $\mathfrak{S}_1$  to a normal operator  $\mathfrak{N}_1$  in a Hilbert space  $\mathfrak{S}$  containing  $\mathfrak{S}_1$  as a subspace. If  $\mathfrak{N}_1$  is such an operator and if  $\mathfrak{D}(\mathfrak{N}_1) \cap (\mathfrak{S} \ominus \mathfrak{S}_1)$  is dense in  $\mathfrak{S} \ominus \mathfrak{S}_1$ , then  $\mathfrak{N}_1$  restricted to  $\mathfrak{D}(\mathfrak{N}_1) \cap (\mathfrak{S} \ominus \mathfrak{S}_1)$  is a formally normal operator  $N_2$  in  $\mathfrak{S}_2 = \mathfrak{S} \ominus \mathfrak{S}_1$ . Moreover the operator  $\mathfrak{N} = N \oplus N_2$  is formally normal in  $\mathfrak{S}$  and satisfies  $N \subset \mathfrak{N} \subset \mathfrak{N}_1 \subset \mathfrak{N}^*$ . The results of §2 are then applied to characterize all such  $\mathfrak{N}_1$ .

A sufficient condition for a normal extension  $\mathfrak{N}_1$  of  $N$  to be such that  $\mathfrak{D}(\mathfrak{N}_1) \cap \mathfrak{S}_2$  is dense in  $\mathfrak{S}_2$  is that the dimension of the space  $\mathfrak{N}(I + N^*\bar{N}^*)$  be finite. In this case it is shown that

$$\mathfrak{N}(I + N^*\bar{N}^*) = \mathfrak{N}(I + \bar{N}^*N^*),$$

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