

Convexity and Pseudoconvexity for complex manifolds.

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1. Introduction. It has long been known that real hypersurfaces in complex Euclidean spaces that are convex or are tubes over convex hypersurfaces on the real part of the space are pseudoconvex in the sense of complex function theory. Indeed, this fact leads to the most practical method for constructing a supply of pseudoconvex hypersurfaces, hence also domains of holomorphy. The purpose of this note is to indicate an extension to the case where a hypersurface of a manifold M has certain convexity properties with respect to an affine connection on M . Precisely, we will prove:

Theorem A. *Let M be a complex manifold with a torsion-free affine connection whose parallel translation leaves the complex structure invariant. Let N be a submanifold of M and, for each $p \in N$, let H_p be the maximal holomorphic subspace of N_p (the tangent space to N at p .) The Levi form of N at p is a symmetric bilinear form $L : H_p \times H_p \rightarrow M_p/N_p$. (M_p is the tangent space to M at p .) The second fundamental form of the submanifold N with respect to the affine connection is a symmetric bilinear form $S : N_p \times N_p \rightarrow M_p/N_p$.*

The basic formula relating L and S is now:

$$(1.1) \quad L(u, v) = S(u, v) + S(Ju, Jv) \quad \text{for all } v \in H_p, \text{ all } p \in N.$$

(J is the tensor field of type $(1, 1)$ on M defining the almost complex structure underlying the given complex manifold structure on M .)

The Levi form is usually defined when N is a hypersurface, so one of our new results is that it also can be defined for arbitrary submanifolds. Similarly, the second fundamental form is usually only defined in case the affine connection is associated with a Riemannian metric.

Of course, the case where N is a hypersurface is the most important for complex function theory. In this case we can read off from 1.1 the relation between convexity and pseudoconvexity. Recall the following definitions:

- a) The hypersurface N is *pseudoconvex* (with respect to the complex structure)

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