

Hamilton's Theory and Generalized Solutions of the Hamilton-Jacobi Equation

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1. Introduction. Let $f(t, x, v)$ be of class C'' and let

$$f_{vv}(t, x, v) > 0 \quad \text{and} \quad f_v(t, x, \pm\infty) = \pm\infty,$$

respectively, for all values of the real variables t, x, v . Let $g(t, x, z)$ be the conjugate function figuring in the Legendre transformation and its inverse

$$\begin{aligned} z &= f_v(t, x, v), & v &= g_z(t, x, z), \\ f(t, x, v) + g(t, x, z) &= vz, \end{aligned}$$

(with t, x as parameters). $g(t, x, z)$ is then defined and also of class C'' for all values of t, x, z .

The classical theory of Hamilton (see the respective chapters in [1] and [2]) furnishes smooth solutions of the Hamilton-Jacobi equation

$$(1.1) \quad J_t + f(t, x, J_x) = 0,$$

by means of the associated variational problem

$$\delta \int_{t_0}^{t_1} g(\tau, x(\tau), \dot{x}(\tau)) d\tau = 0,$$

in the following direct way. Consider the endpoint $t_1, x_1 = x(t_1)$ fixed and the other endpoint $(t_0, x(t_0))$ variable on a fixed curve γ (of class C'') in the plane of (t, x) . The solutions $x(\tau)$ of this variational problem, for various fixed endpoints (t_1, x_1) are the extremals (extremal = stationary curve for fixed endpoints) that meet γ transversally. Suppose that these transversal extremals form a field in some one-sided neighborhood N of γ . Then, for any point (t, x) inside of N , the expression

$$J(t, x) = \int_{t_0}^t g(\tau, x(\tau), \dot{x}(\tau)) d\tau,$$

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