

# *A Geometric Property of the Two-dimensional Harmonic Green's Function*

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If  $D$  is a simply-connected domain in the  $xy$ -plane containing the point at infinity, we denote by  $g(x, y)$  the harmonic Green's function of  $D$  with singularity at infinity, i.e.,  $g(x, y) - \frac{1}{2} \log(x^2 + y^2)$  is harmonic in  $D$ , and  $g(x, y)$  approaches zero if the point  $(x, y)$  approaches the boundary of  $D$  from within. We shall establish the following result.

**Theorem I.** *Let  $D$  be a simply-connected domain in the  $xy$ -plane containing the point at infinity, and denote by  $\Gamma$  the complement of  $D$ . If  $H(\Gamma)$  is the convex hull of  $\Gamma$ , and if  $g(x, y)$  is the harmonic Green's function of  $D$ , then the set  $H(\Gamma) - \Gamma$  is contained in the set whose points satisfy*

$$(1) \quad 0 < g(x, y) \leq \frac{1}{2} \log 2.$$

*The bound  $\frac{1}{2} \log 2$  is the best possible. The set  $H(\Gamma) - \Gamma$  will contain a point  $(x, y)$  at which  $g(x, y) = \frac{1}{2} \log 2$  if, and only if,  $\Gamma$  is a semicircular arc.*

Before we prove this result, we note that it has the following corollary.

*Let  $\Gamma$  be a closed connected plane set of logarithmic capacity  $C$ . If  $H(\Gamma)$  is the convex hull of  $\Gamma$  and  $C_1$  denotes the logarithmic capacity of  $H(\Gamma)$ , then*

$$(2) \quad C_1 < \sqrt{2} C.$$

Indeed,  $C$  is defined by the expansion  $g(x, y) = \frac{1}{2} \log(x^2 + y^2) - \log C + O[(x^2 + y^2)^{-1}]$  for large  $x^2 + y^2$ , and the assertion follows from (1) and the fact that the logarithmic capacity is an increasing domain function, if it is observed that the region bounded by the level line  $g(x, y) = \log r$  has the logarithmic capacity  $rC$ . However, unlike (1), (2) is not a sharp estimate, and the question as to the smallest constant  $K$  for which  $C_1 \leq KC$  remains open. The best lower estimate for  $K$  we can offer is

$$K \geq 3\pi^{-2} 2^{-7/3} \Gamma^3(\frac{1}{3}) = 1.159 \dots$$

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