

A Necessary Condition for the Existence of Best Approximations

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The purpose of this note is to prove the following theorem:

Theorem. *Let B be a Banach space and X a bounded subset of B . In order that for each bounded real-valued function F on X there exist a continuous linear functional $\phi \in B^*$ such that $\Delta(\phi) = \sup \{|F(x) - \phi(x)| : x \in X\} = \inf \{\Delta(\psi) : \psi \in B^*\}$, it is necessary that 0 be in the interior of the closed, balanced, convex hull K of X relative to its linear span $[K]$.*

In the preceding paper by E. W. Cheney and A. A. Goldstein, it was proved that this condition is also sufficient to insure the existence of best approximations even when X is not bounded. We shall actually show that when the condition fails, there is a linear function F on $[K]$ such that

- (a) the restriction of F to X is continuous
- (b) F can be uniformly approximated on X within any desired degree of accuracy by functionals in B^* , but
- (c) the restriction of F to X is not equal to the restriction to X of any functional in B^* .

Let ρ be the norm of B . Suppose K has no interior relative to the ρ -topology on $[K]$. Since K is symmetric, convex, closed, and absorbing in $[K]$, there is a norm σ on $[K]$ such that $K = \{x \in [K] : \sigma(x) \leq 1\}$ (Bourbaki [1, p. 95]). Since K is ρ -bounded, the new topology induced by σ on $[K]$ is finer than the ρ -topology. Because K is ρ -complete, the σ -topology on $[K]$ is complete, so that σ makes $[K]$ into a Banach space (Bourbaki, [1, Chap. I, Cor. of Prop. 8]).

Let us show that $[K]$ cannot be ρ -closed in B . In the contrary case, it would follow from the open mapping theorem that σ and ρ were equivalent norms. (Dunford and Schwartz [2, Theorem II. 2.5]). But this would contradict our hypothesis that K has no interior relative to the ρ -topology on $[K]$.

Thus the ρ -closure L of $[K]$ is actually larger than $[K]$. The inclusion map

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