

A Relation Involving Hankel Transforms with Applications to Boundary Value Problems in Potential Theory

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1. Introduction. One of the classic problems of mathematical physics is that of determining the potential of the electrostatic field due to a circular disk whose potential is prescribed. One method of solving this problem is to determine the charge density σ on the disk and then to calculate the potential at a field point \mathbf{r} by evaluating the integral

$$\int \frac{\sigma(\mathbf{r}') dS'}{|\mathbf{r} - \mathbf{r}'|}$$

over the surface of the disk. In the case in which the prescribed potential of the disk $V(\rho)$ is a function only of the distance ρ from the center of the disk, Beltrami [1] showed that the density of the surface charge is given by the formula

$$(1.1) \quad \sigma(\rho) = -\frac{1}{\pi^2 \rho} \frac{d}{d\rho} \int_{\rho}^{\infty} \frac{x dx}{\sqrt{(x^2 - \rho^2)}} \frac{d}{dx} \int_0^x \frac{y V(y) dy}{\sqrt{(x^2 - y^2)}}, \quad 0 \leq \rho \leq a,$$

where a is the radius of the disk.

The object of this short note is to point out that Beltrami's relation is a special case of a general relation between Hankel transforms. If we denote the operator of Hankel transforms of order ν by \mathcal{H}_{ν} , so that

$$(1.2) \quad \mathcal{H}_{\nu} f(x) = \mathcal{H}_{\nu} \{f(t); x\} = \int_0^{\infty} t f(t) J_{\nu}(xt) dt,$$

we show that the function

$$\mathcal{H}_{\mu} \{ \xi^{\delta} \mathcal{H}_{\nu} f(\xi); \rho \}$$

can be expressed as a double integral involving $f(\rho)$ which is a generalization of the integral occurring on the right hand side of Beltrami's relation (1.1). By taking particular values of the parameters δ , μ , ν occurring in this relation

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