

On the n -dimensional Integral of Marcinkiewicz

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Communicated by A. ZYGMUND

Introduction. Recently (see [2]), E. M. Stein and A. Zygmund proved the following theorem, in which $x = (x_1, \dots, x_n)$ is a point in n -dimensional Euclidean space E^n and likewise $h = (h_1, \dots, h_n)$, $|h| = (\sum_{i=1}^n h_i^2)^{1/2}$, $dh = dh_1 \cdots dh_n$, etc.

Theorem 1. Suppose $F(x) \in L^2(E^n)$ and for each point x belonging to a set $E \subset E^n$, F satisfies the condition

$$(1) \quad F(x+h) + F(x-h) - 2F(x) = o(|h|),$$

as $|h| \rightarrow 0$. Then the necessary and sufficient condition for F to have a total differential almost everywhere in E is that the integral

$$(2) \quad \int_{E^n} \frac{[F(x+h) + F(x-h) - 2F(x)]^2}{|h|^{n+2}} dh$$

be finite almost everywhere in E .

The integral (2) is an n -dimensional analogue of the well-known integral of Marcinkiewicz, defined for functions of a single variable by

$$(3) \quad \int_0^\infty \frac{[f(x+t) + f(x-t) - 2f(x)]^2}{t^3} dt,$$

(See [4] p. 219).

Stein and Zygmund have obtained in [2] a number of results concerning generalizations of (3) and higher derivatives of F , but for functions of several variables such generalizations seem to be quite difficult, and they confine their attention there solely to the case of the first differential. In addition to Theorem 1, they state the following:

Theorem 2. If the function $F(x) = F(x_1, \dots, x_n) \in L^2(E^n)$ and at each point of a set $E \subset E^n$ has a first differential in the metric L^2 (and in particular, if it has an ordinary first differential in E), then the integral (2) exists almost everywhere in E .