

*Tchebycheff Approximation and Related Extremal Problems**

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Part 1. One of the principal maneuvers available for proving existence theorems is to establish that the desired point is where a continuous real-valued functional achieves an extremal on a compact set. If the extremal sought is an infimum, then the function need only be lower semicontinuous. It is convenient to define lower semicontinuity of a functional ϕ to mean that each set of the form $\{x : \phi(x) \leq c\}$ is closed. The crux of an existence proof along these lines would then be the introduction of an appropriate topology in which the functional is lower semicontinuous and the set compact. This idea is applied here to several problems involving convex functionals, particularly the Tchebycheff approximation problem. In part II the characterization of the extremals in these problems is investigated. It is a pleasure to thank R. T. Rockafellar for suggesting improvements.

The terminology used below follows that of the treatise [1] of Dunford and Schwartz, except for retention of the older terms *weak* and *weak** topology.

1. Theorem. *A continuous convex functional defined on a convex set with a nonvoid interior in a linear topological space is the supremum of a family of continuous affine functionals.*

Proof. Let φ be such a functional, G its domain, E the linear space, and R the reals. In $E \oplus R$ consider the upper graph set of φ :

$$K = \{(x, \lambda) : x \in G, \lambda \in R, \lambda \geq \varphi(x)\}.$$

Since φ is upper semicontinuous, K has an interior; since φ is convex, K is convex; since φ is lower semicontinuous, K is closed. By [2, page 72, Corollary], K is the intersection of a family of half spaces H_α , say

$$H_\alpha = \{(x, \lambda) : f_\alpha(x) - c_\alpha \lambda \leq d_\alpha\},$$

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