

# *On the Embeddability of Partially Ordered Halfrings*

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*Communicated by G. BIRKOFF*

**1. Introduction.** In this paper we attack the problem of embedding a partially ordered halfring in a partially ordered ring [2]. Clifford [6] summarized related results for semigroups in his invited address before the American Mathematical Society. We give the pertinent definitions for our theory in Section 2, and in Section 3 we prove the main theorem, namely *necessary and sufficient conditions that a partially ordered halfring is embeddable in a ring*. Examples illustrating many of the points of our theory are given in Section 4.

Our theory is helpful in the study of convex ideals [7] in partially ordered halfrings and Banach halfalgebras [5]. The second author will develop this in a forthcoming paper.

**2. Preliminaries.** For the sake of completeness we repeat the definition of the term *semiring* in the sense stated by one of the authors in an earlier paper [3].

**Definition 1.** A semiring  $S$  is a system consisting of a set  $S$  and two binary operations on  $S$  called addition and multiplication such that (a)  $S$  together with addition is a semigroup; (b)  $S$  together with multiplication is a semigroup; (c) the left and right hand distributive laws  $a(b + c) = ab + ac$  and  $(b + c)a = ba + ca$  hold in  $S$ .

**Definition 2.** A halfring [8] is a semiring which is embeddable in a ring.

**Definition 3.**  $(H, \leq)$  is a partially ordered halfring, briefly a p.o. halfring if and only if (a)  $H$  is a halfring, (b)  $H$  is partially ordered by  $\leq$ , (c)  $s \leq t$  implies that  $s + x \leq t + x$ , and (d)  $s \leq t$ ,  $x \geq 0$  imply that  $sx \leq tx$  and  $xs \leq xt$ .

**Definition 4.** A partial ordering  $\leq$  is said to be strongly additive if and only if  $s \leq t$  is equivalent to  $s + x \leq t + x$ .

**Definition 5.** A partial ordering  $\leq$  is said to be strongly multiplicative if and only if for  $x > 0$ ,  $s \leq t$  is equivalent to  $xs \leq xt$  and  $sx \leq tx$ .

\* This author would like to thank the National Science Foundation for financial support.