

Hermitian Differential Geometry and the Theory of Positive and Ample Holomorphic Vector Bundles

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In his well-known paper [10], Kodaira introduced the concept of a *positive line bundle* over a compact Kähler manifold. He then proved a “vanishing theorem” for the cohomology groups of the associated coherent analytic sheaf and, in [11] Kodaira used this vanishing theorem to show that, if \mathbf{E} is a positive line bundle, then some power $\mathbf{E}^\mu = \mathbf{E} \otimes \cdots \otimes \mathbf{E}$ (μ times) has “sufficiently many sections.” The converse is also true: if \mathbf{E} is *ample*, then \mathbf{E} is positive.

Now, although in [14] and [13] there was some mention of a definition of positivity for general holomorphic vector bundles, it seems that there has never been any explicit attempt to extend Kodaira’s results just mentioned. As a possible exception to this, we mention Grauert’s paper [6] in which he introduced the notion of a weakly positive holomorphic vector bundle.

On the other hand, there has recently been some attention directed to a problem of Nirenberg and Spencer ([15] and [16]), the so-called *rigidity problem for holomorphic embeddings*. Let $X \subset W$ be a germ of a holomorphic embedding with normal bundle \mathbf{N} . Then Nirenberg and Spencer showed that this germ of embedding is formally rigid whenever \mathbf{N} is either positive or negative; in the positive case, uniqueness holds. In [6], Grauert established the actual (*i.e.* convergent) rigidity theorem when \mathbf{N} is weakly negative by his definition. However, Hironaka has given a counter-example to the actual rigidity in case \mathbf{N} is positive in some sense which has not yet been determined. It then seemed that, in order to further discuss the rigidity and related problems, it was necessary to have clearly in mind what is meant by a positive and/or ample holomorphic vector bundle. The motivation for this paper was to clarify these points for utilization in a companion manuscript on the extension and rigidity problems for positive embeddings.

We now briefly discuss the main points which we hope to make. Recall that a holomorphic line bundle \mathbf{E} is said to be positive if a certain Hermitian form $\Theta(\xi, \xi)$ is positive definite, where Θ involves the curvature of \mathbf{E} and ξ is an