

Best Approximation and Convergence Properties of Higher-Order Spline Approximations

J. H. AHLBERG, E. N. NILSON, J. L. WALSH

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Introduction. We published in February, 1963, [1] an abstract indicating various properties of periodic spline curves of odd degree. These properties were direct extensions of our results on best approximation and convergence of periodic cubic splines which had previously been published in detail [2, 3]. Subsequent to the publication of the abstract, deBoor [4] presented the extension of these best approximation properties to non-periodic splines of odd degree. More recently, Schoenberg [5] indicated his independent derivation of these latter results, and related spline theory the work of Sard [6].

The purpose of the present paper is to present the proofs of the results previously announced for the periodic spline approximation of odd degree, including splines on non-uniform meshes. A simple proof of the existence of the spline approximation is given which extends as well to the non-periodic case. In addition, the periodic spline for uniform meshes is studied in some detail, with explicit expressions given for the spline coefficients together with the inverse of the coefficient matrix and its norm.

Best approximation properties. Let a subdivision $0 = x_0 < x_1 < \dots < x_N = L$ be given. A periodic spline $y(x)$ of degree $2n + 1$ for this mesh is defined to be a function of period L which over each interval $x_{i-1} \leq x \leq x_i$ coincides with a polynomial of degree $2n + 1$, so prescribed at the locations x_i that $y(x)$ for all x has continuous derivatives through order $2n$.

Certain extremal properties of the periodic spline $w(x)$ of degree $2n + 1$ are consequences of a minimal property of its $(n + 1)^{\text{st}}$ derivative, which is itself a direct extension of Holladay's [7] result (*cf.* Theorem 2 of Reference 2).

Theorem 1. *Let $G(x)$ and its first $n + 1$ derivatives be continuous, $-\infty < x < \infty$, and have period L . Let $w(x)$ be a periodic spline on the mesh $0 = x_0 < x_1 < \dots < x_N = L$, having degree $2n + 1$ and period L . Assume*

$$(1) \quad w(x_j) = G(x_j) + C, \quad j = 1, 2, \dots, N.$$