

# *Duality in the Cohomology Ring of Transitive Permutation Representations<sup>1</sup>*

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**Introduction.** Let  $(G, X)$  be an arbitrary permutation representation of a finite group  $G$  acting on the left on a finite set  $X$  (see [1] for the cohomology of permutation representations). Let  $R$  be a commutative, associative ring with unit element on which  $G$  acts trivially. The cohomology ring  $H^*(X; G, R)$  of  $(G, X)$  with values in  $R$  has been constructed in [2], and we investigate in the present paper the duality relations in  $H^*(X; G, R)$ .

If  $(G, X)$  is the regular representation of  $G$  (*i.e.*,  $(G, X)$  is transitive and free of fixed points), the classical cohomology theory of groups gives us the duality relations in the ring  $H^*(X; G, Z)(= H^*(G, Z))$ , where  $Z$  is the ring of the rational integers. (See [3], Chapter XII, Section 6.)

Suppose now that  $(G, X)$  is transitive but not necessarily free of fixed points. We denote the order of  $G$  by  $n$  and the number of elements of  $X$  by  $m$ ; then,  $m$  divides  $n$  and we denote the quotient  $n : m$  by  $h$ , and the greatest common divisor of  $h$  and  $m$  by  $d$ . These are the natural integers to study since  $X$  may now be identified with the set of the left cosets of the subgroup  $H = \{\sigma \mid \sigma \varepsilon G, \sigma x_0 = x_0\}$  of  $G$ , where  $x_0$  is an arbitrarily chosen element of  $X$ ;  $G$  acts on these left cosets by left multiplication and  $m = [G : H]$ ,  $h = [H : 1]$  and  $d = (h, m)$ . In particular,  $H$  is a Hall subgroup of  $G$  if and only if  $d = 1$ , which includes the fixed point free case when  $h = 1$ . For general  $d$ , we replace the ring  $Z$  by the ring  $Z[1/d]$  which consists of the rational numbers  $z/d^s$ , where  $z \varepsilon Z$  and  $s = 0, 1, 2, \dots$ . We show that all the usual duality relations are valid in the cohomology ring  $H^*(X; G, Z[1/d])$ , where  $G$  acts trivially on  $Z[1/d]$ . (See Section 6.) In the one extreme, namely when  $H$  is a Hall subgroup of  $G$ , this result is as good as one could hope for because then  $Z[1/d] = Z$ ; in the other extreme, namely when  $h = m$ , this result is without value because the whole cohomology ring  $H^*(X; G, Z[1/d])$  then collapses to zero. (See Remark 6.2.)

The difficulties which must be overcome before duality can be investigated for non transitive permutation representations have been brought out in Remark 4.2. The restriction to transitivity does not occur in the first three sections.

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