

Discrete Harmonic Kernels

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1. Introduction. The finite difference analogue of the Laplace operator has been studied extensively as a means of approximating solutions of the first boundary value problem of potential theory (the Dirichlet problem) for the Laplace equation. In their fundamental paper, Courant, Friedrichs, and Lewy [3] showed that the discrete harmonic approximants converge to the harmonic solution of the Dirichlet problem when the mesh size decreases to zero. They also introduced a discrete analogue of the Green's function which converges to the usual harmonic Green's function with decreasing mesh.

The discrete analogue of the second boundary value problem (the Neumann problem) has not received so much attention due to the inherent difficulties in interpreting the normal direction to the boundary in the language of finite differences on a rectilinear mesh. Inoue [6] did study a discrete analogue of the second boundary value problem and defined a discrete Neumann's function. He did not, however, investigate what happened as the mesh width decreased to zero. Giese [5] established that in a rectangular domain, the solution of the finite difference analogue (discrete harmonic solution) of the Neumann problem converges to the harmonic solution of the Neumann problem with decreasing mesh.

In this study, an orthogonal basis of the space of discrete harmonic functions is used to define a discrete analogue of the Bergman kernel function. It is shown that this reproducing kernel has many of the properties of the harmonic kernel; for example, it turns out that the discrete kernel is the difference between the discrete Neumann's function of Inoue and the discrete Green's function of Courant, Friedrichs, and Lewy. The convergence of the kernel to the harmonic kernel is established for a rectangular domain. It is anticipated that the convergence of the discrete kernel to the harmonic kernel can be established for more general domains by using the abstract properties of the function spaces involved. These results would also shed more light on the behavior of the discrete Neumann's function.

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