

*Submanifolds of Euclidean Spaces**

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1. Introduction. The goal of this paper is to give a positional characterization of submanifolds of the Euclidean spaces, *i.e.*, of subsets of the Euclidean spaces which admit differentiable structures relative to which the identity map is continuously differentiable with no critical points. The technique consists of extending the standard theory of differentials to include functions defined on arbitrary subsets of a Euclidean space. Tangent spaces are defined geometrically and differentials are linear mappings on these tangent spaces. Assumptions of a positional character are then applied to these arbitrary subsets so as to support a proof of the inverse function theorem. Local coordinates then arise as restrictions, to the set, of orthogonal projections onto the tangent spaces followed by isomorphisms of the latter onto Euclidean spaces of appropriate dimension. Submanifolds turn out to be locally compact sets with no isolated points and no “edge-points” whose “tangent spaces slide continuously” and which are “locally connected by almost straight paths.” (The quoted terms are defined in the body of the paper.)

2. Notation. The Euclidean space of dimension n is denoted R^n and its points, also called vectors, are the usual n -tuples $x = (x_1, \dots, x_n)$. The coordinate projection $x \rightarrow x_i$ is denoted by π_i . If f is a function with values in R^n its *coordinate functions* are the compositions $\pi_i \circ f$. When U is open in R^n the partial derivatives of $f : U \rightarrow R^n$ are denoted f_i with

$$(2.1) \quad f_i(x) = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x)}{h},$$

and the Jacobian of $g : U \rightarrow R^k$, the matrix function whose entries are $(\pi_i \circ g)_i$, is denoted by J_g . The Kronecker vector with all coordinates zero except for the i -th which is one is denoted u_i .

3. Tangent spaces and differentials. Let B be a subset of R^n . A unit vector t in R^n is *tangent to B at b* provided there exists a sequence $\{p_m\}$ in $B - (b)$

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