

A Non-Linear Integral Equation and Its Application to Critical Branching Processes

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1. Introduction and summary. Let f be a (probability) generating function, i.e., a power series with non-negative coefficients whose sum is one; and assume that in addition

$$(1.1) \quad f'(1) = 1; \quad f'''(1) < \infty.$$

Let G be a probability distribution function on the real line such that

$$(1.2) \quad G(0-) = 0; \quad G(0) < 1; \quad 1 - G(t) = O(t^{-3}).$$

Let $m = \int_0^\infty t \, dG(t)$, and $b = 2m/f''(1)$. The purpose of this paper is to prove the following result.

Theorem. *If f and G satisfy (1.1) and (1.2), and F is a non-negative, bounded function defined on the non-negative real line which satisfies the equation*

$$(1.3) \quad F(t) = \int_0^t f[F(t-y)] \, dG(y), \quad t \geq 0,$$

then

$$\lim_{t \rightarrow \infty} t[1 - F(t)] = b.$$

The work was motivated by a problem related to branching processes. An object born at time 0 has a life length which is a random variable with distribution G . At the end of its life it is replaced by a random number of objects, the number being n with probability p_n . The new objects reproduce in a similar manner, and the process continues as long as any objects exist. Let $Z(t)$ denote the number of particles existing at time t and let

$$f(s) = \sum_{n=0}^{\infty} p_n s^n,$$

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