

Two Remarks on Sprays

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A spray can be considered either as a system of parametrized curves on the base manifold [1], or as a vector field up on the tangent bundle [2], [4]. We adopt the latter point of view in this note. The first observation we wish to make is that the main theorem of [1] on the one-one correspondence between sprays and dissections can be given a very simple proof in this setting. While unavoidably making use of local coordinates, our proof is nonetheless non-computational and seems more transparent than the previous proof given in [1]. Moreover, it bears out the fact that the most natural connection associated with a given dissection is indeed the torsionless one—a fact dictated by mathematical aesthetics. The second object of this note is to characterize those sprays that come from a metric. Roughly speaking, we require the existence of a horizontal one-form which behaves right under Lie differentiation with respect to the spray and which does not vanish too often. This characterization seems fairly weak and a more subtle one would be highly desirable. The note closes with a few obvious remarks about the uniqueness of the metric.

I. Notations. \mathbf{TM} will always denote the tangent bundle of the d -dimensional manifold M , and $\pi : \mathbf{TM} \rightarrow M$ is the canonical projection. Let $\{x_1, \dots, x_d\}$ be a coordinate system around $m \in M$, then $\{x_i\}$ induces a natural coordinate system $\{x_1, \dots, x_d, v_1, \dots, v_d\}$ about $(m, s) \in \mathbf{TM}$ as follows:

$$x_i(m, s) = x_i(m), \quad v_i(m, s) = sx_i, \quad 1 \leq i \leq d.$$

All indices will run from 1 to d from now on unless otherwise stated. To avoid confusion, we denote the coordinate vector fields of $\{x_i, v_i\}$ by

$$X_i(m, s) \equiv \frac{\partial}{\partial x_i}(m, s), \quad V_i(m, s) \equiv \frac{\partial}{\partial v_i}(m, s).$$

So when we write X_i , we mean $\partial/\partial x_i$ up in \mathbf{TM} , and $\partial/\partial x_i$ will stay below in M as usual. We also use $\mathbf{V}(m, s)$ to denote the vertical space of \mathbf{TM} at (m, s) , *i.e.* $\mathbf{V}(m, s) = \text{span} \{V_i(m, s)\}$. For each $0 \neq \alpha \in \mathbf{R}$, a multiplication diffeo-

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